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ON SOME SPECIAL LAWS OF NON-LINEAR FILTRATION*

G.A. DOMBROVSKII

New laws of the non-linear filtration of an incompressible fluid are proposed (including laws of filtration with a limiting gradient /1/) which enable one, when solving planar, stationary problems, to make use of the apparatus of the theory of functions of a complex variable. Some well-known special cases are considered.

1. The planar stationary filtration of an incompressible fluid is considered. Let $z = x + iy$ be the plane of flow, v be the modulus of the filtration velocity vector, θ be the angle of inclination of the filtration velocity vector to the x-axis, ψ be the stream function, $\varphi = -H + \text{const}$, where H is the head, and let $\Phi(v)$ be a function which characterizes the filtration law /2/. By adopting v and θ as the independent variables, we shall have a system of equations

$$\frac{\partial\varphi}{\partial\theta} = \frac{\Phi^2(v)}{v\Phi'(v)}\frac{\partial\psi}{\partial v}, \quad \frac{\partial\varphi}{\partial v} = -\frac{\Phi(v)}{v^2}\frac{\partial\psi}{\partial\theta}$$

for the functions $\varphi(v, \theta)$, $\psi(v, \theta)$ which can be obtained, for example, from the condition of the integrability of the right-hand side of the differential relationship

$$dz = e^{i\theta}\left[\frac{d\varphi}{\Phi(v)} + \frac{i\,d\psi}{v}\right]$$

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which reflects the meaning of the functions Φ and ψ.

If, a function L is introduced and the independent variable σ, instead of v in accordance with the equalities

$$L = \frac{\Phi(v)}{v}\sqrt{\frac{\Phi(v)}{v\Phi'(v)}}, \quad \sigma = \int \sqrt{\frac{\Phi'(v)}{v\Phi(v)}}\, dv \tag{1.1}$$

then we arrive at the basic system in canonical form

$$\frac{\partial\varphi}{\partial\theta} = L\frac{\partial\psi}{\partial\sigma}, \quad \frac{\partial\varphi}{\partial\sigma} = -L\frac{\partial\psi}{\partial\theta}$$

The relationships

$$-L\frac{d}{d\sigma}\left(\frac{1}{\Phi}\right) = \frac{1}{v}, \quad \frac{d}{d\sigma}\left(\frac{1}{v}\right) = -L\frac{1}{\Phi} \tag{1.2}$$

follow from equalities (1.1) which, in the case of a spherical function $L(\sigma)$, **we shall** consider as a system of ordinary differential equations for determining the functions $1/\Phi(\sigma)$ and $1/v(\sigma)$.

2. Let us adopt the condition

$$L(\sigma) = n^2 \operatorname{cth}^2 m\sigma \quad (m, n = \mathrm{const})$$

In this case the solution of the basic system, as is well-known /3/, can be represented in the form

$$\varphi = \operatorname{Re}\{n(mF - \operatorname{cth} m\sigma F')\} \tag{2.1}$$

$$\psi = \operatorname{Im}\{n^{-1}(mF - \operatorname{th} m\sigma F')\}$$

where F is an arbitrary analytical function of the complex variable $\tau = \sigma - i\theta$. A somewhat different representation is also useful. This differs from (2.1) in the sign of the right-hand side of one of the relationships and in the replacement of the argument of the function F by $\omega = \sigma + i\theta$ ($\omega = \bar{\tau}$).

As a result of the solution of system (1.2), we have

$$\Phi = \frac{Bne^{\sigma}}{Ae^{2\sigma}(\operatorname{th} m\sigma - m) + (\operatorname{th} m\sigma + m)}$$

$$v = \frac{Be^{\sigma}\operatorname{th} m\sigma}{n\left[Ae^{2\sigma}(m \operatorname{th} m\sigma - 1) + (m \operatorname{th} m\sigma + 1)\right]}$$

where A and B are constants of integration.

These functions parametrically define (the parameter is σ) two families of laws of non-linear filtration which are of interest with the arbitrary constants m, n, A and B (the parameters of the families). We obtain the laws apertaining to one family when σ is varied in the neighbourhood of $\sigma = -\infty$ and the laws apertaining to the other family when σ is varied in the right neighbourhood of the point $\sigma = 0$. The curves depicting the laws apertaining to the first family in the $v\Phi$ plane emerge from the origin of coordinates. The equalities $v = 0$, $\Phi = 0$, $d\Phi/dv = n^2$ are satisfied when $\sigma = -\infty$. The laws apertaining to the second family are the laws of filtration with a limiting gradient. The equalities

$$v = 0, \quad \Phi = Bn/[m(1 - A)], \quad d\Phi/dv = 0$$

are satisfied when $\sigma = 0$.

Success in solving non-linear filtration problems by the hodograph method is largely dependent on how simple it is to make the transition from the hodograph variables to the x and y variables of the physical flow plane. The following convenient transition formula, which corresponds to representation (2.1) of the solution of the basic system:

$$z = -P_+(\sigma)e^{-\tau}F'(\tau) - P_-(\sigma)\overline{e^{\tau}F'(\tau)} +$$

$$\frac{m^2 - 1}{B}\left[\int e^{-\tau}F'(\tau)\,d\tau - A\overline{\int e^{\tau}F'(\tau)\,d\tau}\right]$$

$$P_\pm(\sigma) = \frac{e^{\pm\sigma}}{2}\left(\frac{n \operatorname{cth} m\sigma}{\Phi} \pm \frac{\operatorname{th} m\sigma}{nv}\right)$$

is obtained from the differential relationship of paragraph 1.

Hence, the determination of the coordinates of the physical flow plane is reduced to evaluating integrals of functions of the complex variable τ.

The transformation formula, which corresponds to the representation of the solution of the basic system in terms of the function $F(\omega)$, has an analogous form.

Remark. If m is replaced by im and n is replaced by in, we obtain formulae which correspond to the condition $L(\sigma) = n^2 \operatorname{ctg}^2 m\sigma$. It is then possible, in an obvious manner, to write down all the required formulae which correspond to the conditions $L(\sigma) = n^2 \operatorname{tg}^2 m\sigma$, $L(\sigma) = n^2 \operatorname{th}^2 m\sigma$.

3. We will now consider some special cases of the results presented in paragraph 2 and obtained from the condition $L(\sigma) = n^2 \operatorname{cth}^2 m\sigma$.

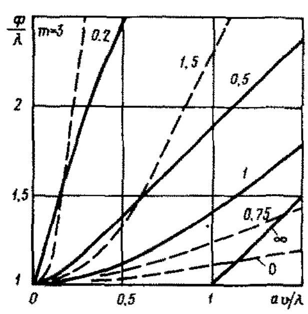
1°. Let us replace F by F/m and B by mB and make m tend to infinity. In the limit, we arrive at formulae which follow from the condition $L = n^2 = \text{const}$. A method of solving non-linear filtration problems, which is, in essence, based on this condition, was proposed in /4/ (σ varies in the neighbourhood $\sigma = -\infty$). The case when $L = \text{const}$ has also been considered in /5/, where a law of filtration, which is different from that adopted in /4/, was additionally indicated.

2°. Let us replace n by mn and make m tend to zero. In the limit, we obtain all the formulae for the case when $L(\sigma) = n^2/\sigma^2$. An investigation of the laws of filtration for this case has been carried out in /5/.

3°. Let $A = 0$, $B = \lambda m/n$, $n^2 = a$. When $\sigma \geqslant 0$, we shall have the well-known /6, 7/ family of filtration laws with a limiting gradient

$$\frac{\Phi}{\lambda} = \frac{me^\sigma}{\operatorname{th} m\sigma + m}, \quad \frac{av}{\lambda} = \frac{me^\sigma}{\operatorname{cth} m\sigma + m} \tag{3.1}$$

Using these formulae, the dependence of Φ/λ on av/λ are depicted by the solid lines in the figure for several values of the parameter m. The case when $m = 1$, which is, in particular, characterized by a simple transformation formula, was considered for the first time in /8/. The case when $m = \infty$ is also characteristic. In the limit as $m \to \infty$ we arrive at the formulae for the method proposed in /9/. In order to obtain expression for φ and ψ, as a result of passing to the limit and the transformation formula, it is first necessary to replace F by F/m.



4°. Let us put $A = 0$, $B = \lambda m/n$, $n^2/m^2 = a$. We obtain relationships, which differ from (3.1) in the replacement of the factor m by m^{-1} in the second equality. The curves, calculated using these formulae for different m, are shown in the figure by the broken lines. When $m = 0$, we have the law

$$\frac{\Phi}{\lambda} = \frac{e^\sigma}{\sigma + 1}, \quad \frac{av}{\lambda} = \sigma e^\sigma$$

which has been applied to the solution of some filtration problems with a limiting gradient in /10-12/. The curve, which depicts this law, has a point of inflection with the coordinates $av/\lambda = \sqrt{e}/2$, $\Phi/\lambda = 2\sqrt{e}/3$.

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THE FIELD OF THE POINT SOURCE OF INTERNAL WAVES IN A HALF-SPACE WITH A VARIABLE BRUNT-VAISALA FREQUENCY*

V.A. BOROVNIKOV

Green's function is constructed for the equation of the internal waves in the half-space $z>0$ with a square of the Brunt-Vaisala frequency which is linear with respect to z.

1. Formulation of the problem. The generalized solution $\Gamma(t, \sqrt{x^2+y^2}, z, z_0)$ of the equation

$$L\Gamma = \left(\frac{\partial^2}{\partial t^2}\Delta + B^2 z \Delta_h\right)\Gamma = 0 \tag{1.1}$$

$$\Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad \Delta = \Delta_h + \frac{\partial^2}{\partial z^2}$$

with the initial and boundary conditions

$$\Gamma = 0, \quad \partial\Delta\Gamma/\partial t = \delta(x)\,\delta(y)\,\delta(z-z_0) \quad (t=0) \tag{1.2}$$

$$\Gamma = 0 \quad (z=0) \tag{1.3}$$

is considered in the half-space $z>0$.

It is obvious that, when this function is extended to zero at $t<0$, it satisfies the equation

$$L\Gamma = \delta(t)\,\delta(x)\,\delta(y)\,\delta(z-z_0) \tag{1.4}$$

that is, it is Green's function for the internal wave equation when the square of the buoyancy frequency $N^2(z) = B^2 z$; $B = \text{const}$.

The approximate expression for Γ has the form /1/ (J_0 is a Bessel function and v is an Airy function)

$$G(t, r, z, z_0) = -\frac{1}{\pi^2 B}\int_0^\infty \sigma^{1/3}\,d\sigma \int_0^\infty U\,d\omega \tag{1.5}$$

$$U = v(\sigma^{2/3}(\omega^2 - z))\,v(\sigma^{2/3}(\omega^2 - z_0))\sin B\omega t J_0(\sigma\omega r) \tag{1.6}$$

Since $v(\xi)$ satisfies Airy's equation, it can be shown that the function U and, together with it, also G is an exact solution of Eq.(1.1). The approximate nature of the function G manifests itself in the fact that it does not satisfy the boundary condition (1.3) while the second condition in (1.2) is satisfied with an accuracy up to a smooth term Ψ:

$$\frac{\partial}{\partial t}\Delta G|_{t=0} = \delta(x)\,\delta(y)\,\delta(z-z_0) + \Psi(r, z, z_0)$$

It is natural to assume that the exact Green's function also has the form of (1.5), (1.6) where, however, the product of the Airy functions should be replaced by any other combination of solutions of Airy's equation for the same arguments. The condition regarding the symmetry of Γ with respect to z, z_0, the boundary conditions (1.3) and, finally, the requirement that

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